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Note

Semi-implicit Reduced Magnetohydrodynamics

Recently Harned and Kerner [1] have applied the semi-implicit algorithm [2] to the full magnetohydrodynamics (MHD) equations. However, the reduced MHD equations [3, 4] are still frequently used to describe low beta, large aspect ratio tokamaks. In this note we apply the semi-implicit algorithm to the reduced MHD equations.

The reduced MHD equations [3, 4] are:

$$\frac{\partial \psi}{\partial t} + (V \cdot \nabla)\psi = \eta j + \frac{\partial \phi}{\partial z}$$

$$\frac{\partial U}{\partial t} + V \cdot \nabla U = \mathbf{B} \cdot \nabla j$$

$$j = \Delta \psi, \qquad \phi = \Delta^{-1} U$$

$$\mathbf{B} = \hat{z} + \hat{z} \times \nabla \psi, \qquad \mathbf{V} = \hat{z} \times \nabla \phi.$$

We consider a cylindrical geometry, (r, θ, z), with periodicity assumed in θ and z. An evolution equation at the boundary for either ϕ or $\mathbf{n} \cdot \nabla \phi$, where $\mathbf{n}$ is the normal to the boundary must be given. Both the current and the vorticity are defined only in the interior of the domain. In [3], an evolution equation for $n \cdot \nabla \phi$ is given for the free boundary problem. We concentrate on the fixed boundary problem where the normal velocity to the boundary is zero. This implies that ϕ and ψ are constant on the boundary.

The two standard numerical methods for this system are an explicit predictor corrector advance of the ideal reduced equations [5] or a partially implicit time-centered scheme [6]. In [6], a two-by-two block diagonal system is solved treating all equilibrium quantities implicitly. Following [1], we identify the wave which can violate the Courant–Friedrichs–Levy condition and subtract a model term based on the constant coefficient wave equation from each side of the exact corrector advance. For a uniform poloidal magnetic field in a slab geometry, the shear Alfven wave does not propagate and satisfies $\psi_{tt} = -(k \cdot B_0)^2 \psi$. Thus our corrector step is based on the equation

$$\psi_{tt} + (k \cdot B_0(r))^2 \psi = L(\psi) + (k \cdot B_0(r))^2 \psi,$$

where $L(\psi)$ is the exact nonlinear operator.

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Our scheme is a modification of the predictor corrector algorithm

$$\psi_k^* = \psi_k + \theta\, dt\left([\psi, \phi]_k + \frac{\partial \phi_k}{\partial z}\right)$$

$$U_k^* = U_k + \theta\, dt\left([\psi, j]_k + \frac{\partial j_k}{\partial z} - [\phi, U]_k\right)$$

$$\phi_k^* = \Delta^{-1} U_k^*$$

$$\psi_k^{t+dt} = \psi_k^t + g(r, k)\, dt\left([\psi^*, \phi^*]_k + \frac{\partial \phi_k^*}{\partial z}\right)$$

$$\bar{\psi}_k = \frac{\psi_k^{t+dt} + \psi_k^t}{2}$$

$$\bar{j}_k = \Delta \bar{\psi}_k$$

$$U_k^{t+dt} = U_k^t + dt\left([\bar{\psi}, \bar{j}]_k + \frac{\partial \bar{j}_k}{\partial z} - [\phi^*, U^*]_k\right),$$

where $[A, B] \equiv \hat{z} \cdot \nabla A \times \nabla B$.

The semi-implicit factor $g(r, k)$ is based upon a Fourier decomposition about a cylindrical equilibrium

$$g(r, k) = (1 + dt^2 f\, \mathbf{k} \cdot \mathbf{B}_0(r)^2)^{-1},$$

where k is the Fourier mode number. The traditional predictor corrector scheme corresponds to $g(r, k) = 1$. Thus a single division by the semi-implicit factor $g(r, k)$ converts an explicit predictor corrector scheme to a semi-implicit scheme. We note that this second-order artificial damping is proportional to the mode number and vanishes at the rational magnetic surfaces. Since the physical eigenfunctions are concentrated near the $k \cdot B_0(r) = 0$ surfaces, they are only weakly damped. At the boundaries, where numerical instabilities often occur, the damping factor $g(r, k)$ will be large.

In a uniform magnetic field, the Kreiss amplification matrix for the ideal reduced MHD scheme is

$$G(k) = \begin{bmatrix} 1 - \theta y & \dfrac{iy}{a} \\ ia\left(1 - \dfrac{\theta}{2} y\right) & 1 - \dfrac{y}{2}, \end{bmatrix}$$

where $a = \mathbf{k} \cdot \mathbf{B}_0 / \rho^{1/2}$ and $y = a^2/(1 + f a^2)$. The eigenvalues are

$$\lambda = 1 - \left(\frac{\theta}{2} + \frac{1}{4}\right) y \pm \frac{\sqrt{(\theta + 1/2)^2 y^2 - 4y}}{2}.$$

The argument of the square root is always negative if $f > (\theta + \frac{1}{2})^2/4$. In this case, $|\lambda|^2 = 1 + (\frac{1}{2} - \theta)\, y$. Thus the semi-implicit predictor corrector scheme is unconditionally stable in uniform magnetic fields for $\theta \geqslant \frac{1}{2}$ and $f \geqslant (\theta + \frac{1}{2})^2/4$. For $\theta = \frac{1}{2}$, it is also second-order accurate in the ideal MHD terms. Following [1], the resistive diffusion is done separately.

We have implemented this algorithm on the cylindrical reduced MHD code of [5]. The implementation is two lines long and consists of a single division by the semi-implicit factor $g(r, k)$.

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